# Game theory

## choice under uncertainty

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# Outline

# What is a game?

- A game is a model that captures strategic interaction between players.
- Strategic interaction means that the utility of a player depends on both the decisions taken by the player and the rivals.
- Those decision affect the utility of the other and the environment in which choice are made.

# Why do we use games?

- Decision theory is not sufficient to model markets in which there is strategic interaction between buyers and sellers
- Those markets are classified as imperfectly competitive:
    - market power and its abuse

# Formally

A **one-shot** game in **normal form** can be represented by a table:

- Set of players
- Set of feasible strategies for each player
- Pay off function (of the strategy profile) for each player

# Example A

*Both Chiara and Alessandro need to decide whether to go to the cinema or to the theatre.*

- We have 2 players: Chiara and Alessandro
- The strategies of Chiara are: "Cinema", "Theatre"
- The strategies of Alessandro are: "Cinema", "Theatre"

# Example A

| | | Chiara | |
|---|---|---|---|
| | | Cinema | Theatre |
| **Alessandro** | Cinema | 10,10 | 2,1 |
| | Theatre | 1,2 | 5,5 |

# Example B

*Both Sara and Virginia have decided to open a bar. If both of them decide to open it in the centre of Milan, the expected revenue is 100 € each; if both of them decide to open it in the centre of Monza the expected revenue is 75 € each; if one decides to open it in Milan and the other decides to open it in Monza, the former gets 200€ and the latter gets 150€*

# Example B

Who are the two players?

| | | Sara | |
|---|---|---|---|
| | | | |
| Virginia | | | |
| | | | |

What are the strategies of Sara?

# Example B

| | | Sara | |
|---|---|---|---|
| | | MI | MB |
| Virginia | | | |
| | | | |

What are the strategies of Virginia?

# Example B

| | | Sara | |
|---|---|---|---|
| | | MI | MB |
| Virginia | MI | | |
| | MB | | |

What are the pay offs?

# Example B

| | | Sara | |
|---|---|---|---|
| | | MI | MB |
| **Virginia** | MI | 100, 100 | 200, 150 |
| | MB | 150, 200 | 75, 75 |

# Strategic positioning

2 contrasting effects:

- direct effect: locating closer means steal demand from the rival
- strategic effect: locating closer means stronger price competition
- under some assumptions the latter prevails

### Examples

- TV platform
- shops

# Outline

# Outline

2. Main notions
   - Best response
   - Dominant equilibrium
   - Nash equilibrium
   - Prisoner Dilemma
   - Dynamic and extensive form

# Definition

The best response is the strategy that gives a player the highest pay offs given the decision of the rival(s).

# Intuition

If you anticipate that your rival will play a certain strategy, you can compare your own pay offs for any feasible strategy that you can play

# Example A

| | | Chiara | |
|---|---|---|---|
| | | Cinema | Theatre |
| **Alessandro** | Cinema | 10,10 | 2,1 |
| | Theatre | 1,2 | 5,5 |

If Chiara knows that Alessandro will go to the Cinema, what is she going to do?

She wil go to the Cinema as well. Why?

Because

$$10 > 1$$

# Example B

| | | Sara | |
|---|---|---|---|
| | | MI | MB |
| **Virginia** | MI | 100, 100 | 200, 150 |
| | MB | 150, 200 | 75, 75 |

If Sara knows that Virginia is locating in Milan, where will she decide to open her bar? Why?

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# Rationality and common knowledge of rationality

- rationality: the more is better
- common knowledge of rationality: each player believes that the other players are rational and that they believe that the other players are rational and that...

# Dominant strategy

For a player a dominant strategy is one that is preferred to any other strategy for any decision of the rivals

# Example C

*Matteo and Luca are going to decide how to get to school. The game is the following*

| | | **Matteo** Foot | **Matteo** Bike |
|---|---|---|---|
| **Luca** | Foot | 2, 5 | 3, 10 |
| **Luca** | Bike | 9, 8 | 7, 11 |

- Interpret the game
- What is a dominant strategy for Luca?
- What is a dominant strategy for Matteo?

# Dominant Equilibrium

If all players have a dominant strategy, the corresponding strategy profile is a dominant equilibrium

# Example C

| | | Matteo | |
|---|---|---|---|
| | | Foot | Bike |
| **Luca** | Foot | 2, 5 | 3, 10 |
| | Bike | 9, 8 | 7, 11 |

- What is the dominant equilibrium?

# Example B

| | | Sara | |
|---|---|---|---|
| | | MI | MB |
| **Virginia** | MI | 100, 100 | 200, 150 |
| | MB | 150, 200 | 75, 75 |

- What is a dominant strategy for Sara?
- What is a dominant strategy for Virginia?
- What is the dominant equilibrium?

# The dominant equilibrium doesn't always exist

- It is based on strong assumptions;
- It doesn't always allow us to solve a game

# Outline

2 Main notions

- Best response
- Dominant equilibrium
- Nash equilibrium
- Prisoner Dilemma
- Dynamic and extensive form

# Further assumptions are required

- consistence of believes
- expectations on the other decisions must be correct

# Definition

A Nash Equilibrium is a strategy profile such that each player is best responding to the other

# Intuition

At the Nash Equilibrium no player has a unilateral incentive to deviate (given the decision of the rivals)

# Example A

| | | Chiara | |
|---|---|---|---|
| | | Cinema | Theatre |
| **Alessandro** | Cinema | 10,10 | 2,1 |
| | Theatre | 1,2 | 5,5 |

What is/are the Nash Equilibrium?

# Example A

| | | Chiara | |
|---|---|---|---|
| | | Cinema | Theatre |
| **Alessandro** | Cinema | **10,10** | 2,1 |
| | Theatre | 1,2 | **5,5** |

$$NE = \{(C, C), (T, T)\}$$

# Example B

| | | **Sara** | |
|---|---|---|---|
| | | MI | MB |
| **Virginia** | MI | **100, 100** | 200, 150 |
| | MB | 150, 200 | 75, 75 |

Is it $(MI, MI)$ a Nash Equilibrium?

# Example C

| | | **Matteo** | |
|---|---|---|---|
| | | Foot | Bike |
| **Luca** | Foot | 2, 5 | 3, 10 |
| | Bike | 9, 8 | 7, 11 |

What is the Nash Equilibrium?

# Example C

| | | **Matteo** | |
|---|---|---|---|
| | | Foot | Bike |
| **Luca** | Foot | 2, 5 | 3, **10** |
| | Bike | **9**, 8 | **7**, **11** |

$$NE = \{(B, B)\}$$

# Example D

| | | j: a | j: b |
|---|---|---|---|
| **i** | a | 2, 5 | -1, 6 |
| | b | 0, 0 | 8, 3 |

What is the Nash Equilibrium?

# Example D

**j**

| | | a | b |
|---|---|---|---|
| **i** | a | **2**, 5 | -1, **6** |
| | b | 0, 0 | **8, 3** |

$$NE = \{(b, b)\}$$

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2. Main notions
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# An online example

You are paired with a classmate.

| | | B: COOPERATE | B: DEFECT |
|---|---|---|---|
| A | COOPERATE | 2, 2 | 0, 4 |
| | DEFECT | 4, 0 | 1, 1 |

# An example

| | | **B** | |
|---|---|---|---|
| | | confess (C) | do not confess (N) |
| **A** | confess (C) | 2, 2 | 8, 1 |
| | do not confess (N) | 1, 8 | 4, 4 |

What is the Nash Equilibrium?

$$NE = \{(C, C)\}$$

Why isn't $(N, N)$ a NE?

Because there is a unilateral incentive to deviate!

# Coordination Failure

The prisoner dilemma is an example of coordination failure. The fact that decisions are driven only by one's own interest lead to a suboptimal outcome

# Pareto dominance

The cooperative outcome is a Pareto equilibrium: there is no other strategy profile that would make any player better off leaving the other at least as well off

# Outline

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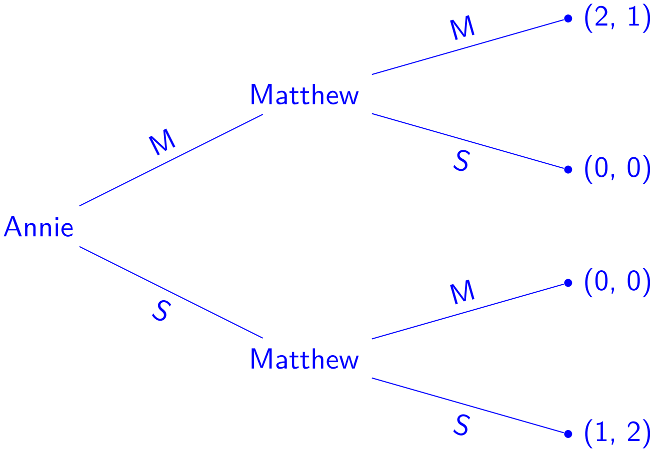
# Tree representation

*Anna and Martina have to decide whether to go to the sea or to the mountains. Anna decides first.*



# Normal form representation

| | | **Matthew** | | | |
|---|---|---|---|---|---|
| | | MM | MS | SM | SS |
| **Annie** | M | $2, 1$ | $2, 1$ | $0, 0$ | $0, 0$ |
| | S | $0, 0$ | $1, 2$ | $0, 0$ | $1, 2$ |

What are the Nash Equilibria?

Is $(S, SS)$ reasonable (*credible*) given the dynamic of the game?

Playing always $S$ is a non credible threat

# Subgame Perfect Nash Equilibria

- A strategy profile such that it is NE of any subgame
- We can solve by backward induction

# Outline

# Outline

# 1st experiment

Choose between 3 alternatives:

- You receive 1000€
- I toss a coin:
  - If head, you win 2000€
  - If tail, you win 0€
- You are indifferent

# Saint Petersbourg Paradox

- Nicolas Bernoulli (18th century)
- Toss a fair coin:
  - The game ends the first time the coins comes up with Head
  - Pays $2^n$ ducats (at the n-th trial)
- How much would you pay *at most* to enter (perceived fair price)?

### Example

- If H at the first time you get 2 ducats
- If TTTH you get $2^4 = 16$ ducats

# Outline

3. Lottery
   - Some experiments
   - Definitions

# Lottery

- $$L = \{c_1, p_1; \ldots; c_n, p_n\}$$
- $c_i$: outcomes/pay offs
- $p_i \in [0, 1]$: probabilities

# Expected Value

$$EV(L) = \sum_{i=1}^{n} c_i p_i$$

# 1st experiment - analysis

We have 2 lottey

- sure choice:
    - $L^* = \{c_1 = 1000, p_1 = 1\}$
    - $EV(L^*) = 1000 \cdot 1 = 1000$
- risky alternative:
    - $L^{**} = \{c_1 = 2000, p_1 = 0.5; c_2 = 0, p_2 = 0.5\}$
    - $EV(L^{**}) = 2000 \cdot 0.5 + 0 \cdot 0.5 = 1000$
- risk lover or risk averse or risk neutral

# Saint Petersbourg Paradox - analysis

We have one lottery:

$$L = \{c_1 = 2, p_1 = \frac{1}{2}; c_2 = 2^2, p_2 = \frac{1}{2^2}; \dots\}$$

$$EV = \sum_{i=1}^{+\infty} 2^i \left(\frac{1}{2}\right)^i = \sum_{i=1}^{+\infty} 1 \to +\infty$$

# Expected Utility

- Daniel Bernoulli proposed $u = \log(\cdot)$
- Principle of decreasing marginal utility
- $$EU(L) = \sum_{i=1}^{n} u(c_i)p_i$$
- The certain equivalent $c_L$ is defined as

$$EU(L) = EU(\{c_L, 1\})$$

- The risk premium (increase in risk aversion) is

$$\pi_L = EV(L) - c_L$$

# Compare lotteries

- we can use the expected utility or analogously the certain equivalent (monotonicity)

# Insurance problem

- $L = \{c_1 - a\delta, 1 - p; c_2 + a - a\delta, p\}$
- $a$ is the unit that reimburse 1 € in case of damage
- $\delta$ is the premium paid per unit
- $a\delta$ is the coverage
- if $\delta = p$ the insurance is actuarially fair

# Bibliography


- Jeffrey M. Perloff (2017). Microeconomics, Theory and Applications with Calculus, Pearson
- Paul Belleflamme, Martin Peitz (2015). Industrial Organization, Cambridge